Hale School

Mathematics Specialist

Test 1 --- Term 1 2018

Complex Numbers

Name: Ms Poly Nomial

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Instructions:

- **CAS calculators are NOT allowed**
- **External notes are not allowed**
- **Duration of test: 45 minutes**
- **Show your working clearly**
- **Use the method specified (if any) in the question to show your working (Otherwise, no marks awarded)**
- **This test contributes to 7% of the year (school) mark**

All arguments must be given using principal values.

1. **[3, 4 = 7 marks]**

Give exact expressions for each of the following in the form $a+bi$:

(a) $\dfrac{\overline{1-i}}{(2+i)^2} = \dfrac{1+i}{3+4i}$ ✓ simplifies correctly

$= \dfrac{(1+i)(3-4i)}{(3+4i)(3-4i)}$ ✓ Multiply by complex conjugate.

$= \dfrac{7-i}{25}$

$= \dfrac{7}{25} - \dfrac{1}{25}i$ ✓ sol$^{\underline{n}}$ in correct form

(b) $\left(1-\sqrt{3}i\right)^5 = \left(2 \operatorname{cis} -\frac{\pi}{3}\right)^5$ ✓ converts to polar

$= 32 \operatorname{cis}\left(\frac{-5\pi}{3}\right)$ ✓ applies de Moivres

$= 32 \operatorname{cis} \frac{\pi}{3}$ ✓ states in terms of principal

$= 16 + 16\sqrt{3}i$ ✓ a+bi form

OR (please don't do this way!)

$= 1^5 - 5.1^4\sqrt{3}i + 10.1^3(\sqrt{3}i)^2 - 10.1^2(\sqrt{3}i)^3 + 5(\sqrt{3}i)^4 - (\sqrt{3}i)^5$

$= 1 - 5\sqrt{3}i - 30 + 30\sqrt{3}i + 45 - 9\sqrt{3}i$

$= 16 + 16\sqrt{3}i$

✓ Expands binomial
✓ Calculates complex powers correctly
✓ Calculates $\sqrt{3}$ powers correctly
✓ simplifies correctly.

2. **[4, 2 = 6 marks]**

(a) Use de Moivre's theorem to find all the exact solutions to the equation

$z^4 = i$ in polar form

$z^4 = \text{cis}\frac{\pi}{2}$ ✓ converts to polar

$z = \text{cis}\left(\frac{\pi}{2} + 2\pi k\right)^{\frac{1}{4}}$

$z = \text{cis}\left(\frac{\pi}{8} + \frac{\pi k}{2}\right) \quad k \in \mathbb{Z}$ ✓ applies deMoivre's

$z_0 = \text{cis}\frac{\pi}{8}$ ✓ Solutions correct in polar form

$z_1 = \text{cis}\frac{5\pi}{8}$

$z_2 = \text{cis}\frac{-3\pi}{8}$ ✓ principal arg

$z_3 = \text{cis}\frac{-7\pi}{8}$

(b) Suppose α and β are two distinct roots of $z^n = i$, where n is a positive integer. Explain why $|\alpha + \beta| < 2$.

$|\alpha + \beta| \leq |\alpha| + |\beta|$ $|z_n| = 1$, α & β in different directions

$|\alpha + \beta| < 1 + 1$

$|\alpha + \beta| < 2$ ✓ triangular inequality (or reasoning involving direction)

✓ $|z_n| = 1$

3. **[3, 3 = 6 marks]**

Sketch the following loci on the complex planes provided:

(a) $\operatorname{Re}(iz+1) = \operatorname{Im}(z)$

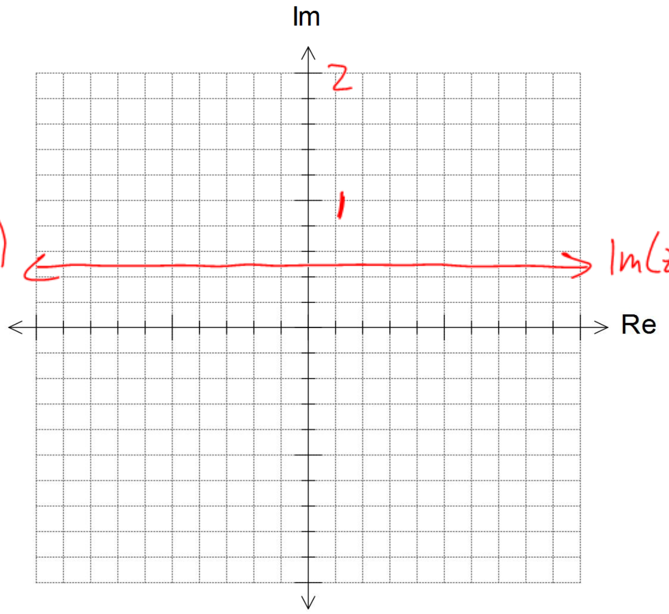
let $z = x + iy$

$\operatorname{Re}(i(x+iy)+1) = \operatorname{Im}(x+iy)$

$-y+1 = y$

$y = \frac{1}{2}$

$\operatorname{Im}(z) = \frac{1}{2}$



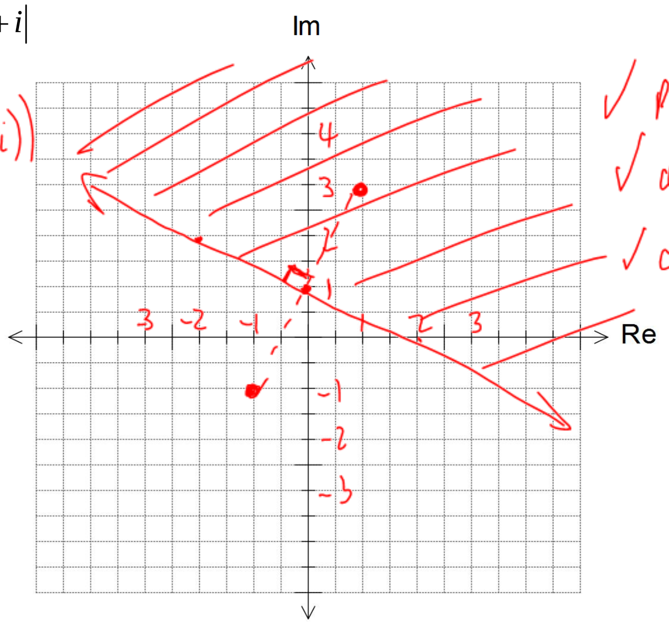
✓ sub $x+yi$

✓ equation

✓ correct plot

(b) $|z-(1+3i)| \le |z+1+i|$

$|z-(1+3i)| \le |z-(-1-i)|$



✓ points plotted

✓ draws the bisector

✓ correct shading.

4. **[4 marks]**

Given $z = (2a+3i)^3$ and $a \in R^+$, find value(s) of a such that $\arg z = 135°$.

$\arg\left(\left(2a+3i\right)^3\right) = 135$ ✓ correct statement

$\arg\left(2a+3i\right) = 45, \; 165, \; -75°$ ✓ Determines $45°$

$\times$ (a +ive) $\times$ ($3i$)

Since $a \in R^+$ $\arg(2a+3i) = 45$

$\frac{3}{2a} = \tan^{-1}(45)$ ✓ calculates arg

$3 = 2a$

$\underline{a = \frac{3}{2}}$. ✓ $a = \frac{3}{2}$

5. **[4 marks]**

Describe the locus of points defined by the equation $|z - i| = 2|z+1|$.

$|x + (y-1)i| = 2|x+1+yi|$

$\sqrt{x^2 + (y-1)^2} = 2\sqrt{(x+1)^2 + y^2}$

$x^2 + y^2 - 2y + 1 = 4[x^2 + 2x + 1 + y^2]$ ✓ square correctly.

$x^2 + y^2 - 2y + 1 = 4x^2 + 8x + 4 + 4y^2$

$0 = 3x^2 + 3y^2 + 8x + 2y + 3$

$0 = x^2 + y^2 + \frac{8}{3}x + \frac{2}{3}y + 1$

$0 = \left(x + \frac{4}{3}\right)^2 - \frac{16}{9} + \left(y + \frac{1}{3}\right)^2 - \frac{1}{9} + 1$

$\frac{8}{9} = \left(x + \frac{4}{3}\right)^2 + \left(y + \frac{1}{3}\right)^2$ ✓ indicates circle.

Circle centre $\left(-\frac{4}{3}, -\frac{1}{3}\right)$ radius $\frac{2\sqrt{2}}{3}$ ✓ radius

✓ centre

6. **[4, 4 = 8 marks]**

(a) The polynomial $x^3 + ax + b$ has a factor of $x+2$ and a remainder of -60 when divided by $x-2$. Determine the values of a and b.

$P(-2) = (-2)^3 + a(-2) + b$

$0 = -8 - 2a + b$

$8 = -2a + b \quad \ldots (1)$

$P(2) = 2^3 + a(2) + b$

$-60 = 8 + 2a + b$

$-68 = 2a + b \quad \ldots (2)$

eq^n (1) + eq^n (2) $\quad -60 = 2b$

$\underline{b = -30}$

$\therefore \quad \underline{a = 19}$

✓ Use $P(-2) = 0$ to determine eq$^{\underline{n}}$

✓ Use $P(2) = -60$ to determine eq$^{\underline{n}}$

✓ $a = 19$

✓ $b = -30$

(b) One root of $P(z) = z^3 + az^2 + 3z + 9$ is purely imaginary. If a is real, find a and hence factorise $P(z)$ into linear factors.

let $P(ki) = 0$

$(ki)^3 + a(ki)^2 + 3(ki) + 9 = 0$

$-k^3 i - ak^2 + 3ki + 9 = 0$

✓ $P(ki) = 0$

Im: $-k^3 + 3k = 0$

$k(-k^2 + 3) = 0$

$k^2 = 3$

$k = \pm\sqrt{3}$

✓ Equates Im

Re: $-ak^2 + 9 = 0$

$-a(\sqrt{3})^2 + 9 = 0$

$\underline{a = 3}$

✓ determines $a = 3$

$\therefore P(z) = (z^2 + 3)(z + 3)$

$P(z) = (z - \sqrt{3}i)(z + \sqrt{3}i)(z + 3)$

✓ factorises into linear

7. **[4, 5 = 9 marks]**

Show that if $z = cis\theta$ then:

a) $z^n - \frac{1}{z^n} = 2i\sin n\theta$

$$
\begin{aligned}
LHS &= z^n - \frac{1}{z^n} \\
&= (cis\theta)^n - \frac{1}{(cis\theta)^n} && \checkmark \text{ sub } z = cis\theta \\
&= (cis\theta)^n - (cis\theta)^{-n} \\
&= cis\,n\theta - cis(-n\theta) && \checkmark \text{ deMoivre Theorem} \\
&= \cos n\theta + i\sin n\theta - [\cos(-n\theta) + i\sin(-n\theta)] && \checkmark \text{ expand } cis \\
&= \cos n\theta + i\sin n\theta - [\cos n\theta - i\sin n\theta] \\
&= 2i\sin n\theta && \checkmark \text{ recognises } \cos(-n\theta) = \cos n\theta,\ \sin(-n\theta) = -\sin(n\theta) \\
&= RHS
\end{aligned}
$$

b) Use the previous result to show that $\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$

$$(2i\sin\theta)^3 = \left(z - \frac{1}{z}\right)^3$$ ✓ raises power 3

$$(2i\sin\theta)^3 = z^3 - 3z^2\frac{1}{z} + 3z\frac{1}{z^2} - \frac{1}{z^3}$$ ✓ expands

$$-8i\sin^3\theta = z^3 - \frac{1}{z^3} - 3\left(z - \frac{1}{z}\right)$$ ✓ group correctly

$$-8i\sin^3\theta = 2i\sin 3\theta - 3(2i\sin\theta)$$ ✓ substitutes

$$-8i\sin^3\theta = 2i\sin 3\theta - 6i\sin\theta$$

$$\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$$ ✓ shows result